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The q-Hermite polynomial and the representations of Heisenberg and quantum Heisenberg algebras

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Abstract. We introduce a pair of canonical-conjugate q-deformed operators D, X and discuss the relations between the deformed operators D, X and q-series. The realizations of some Lie symmetries, Heisenberg and quantum Heisenberg algebras are given in operators D and X. We show that the q-analogous Hermite polynomials are representations of Heisenberg and quantum Heisenberg algebras realized in this way. When q is a root of unity, the properties of the q-analogous Hermite polynomials are also discussed.

1. Introduction

Recently a lot of attention has been paid by physicists and mathematicians to the q-analogous special functions, because of their importance to integrable models, quantum groups and the Yang–Baxter equations. These q-special functions, such as the q-analogous Hermite polynomial, can be obtained from q-series when the parameters of the latter are properly fixed.

The topic of q-series [1] is a century old and has intensive relations with other fields of mathematics such as number theory, classical analysis, combinatorics, additive number theory and Lie algebras. It is remarkable that this elegant mathematical invention has been found useful in physical theories such as the lattice field theories [2] and the hard hexagon model [1, 3] in statistical physics. The q-series naturally arises in Baxter's solution to the hard hexagon model.

In this paper, we introduce a pair of canonical-conjugate q-deformed operators D and X and show that it is convenient to discuss q-series by these q-deformed operators. As an example of the actions of operator-valued functions on a constant, the Biedenharn exponential function $\exp_q(\alpha x)$ as well as $\sinh(\alpha x)$ and $\cosh(\alpha x)$ are obtained naturally. We show that the representations of Lie groups can also be given by means of the operators D and X and demonstrate that the q-analogous Hermite polynomials can be representations of Heisenberg and quantum Heisenberg algebras.

This paper is organized as follows. In section 2, the operators X and D are introduced as q-deformations of the operators x and ∂. The relationship of these deformed operators with q-series is also discussed. In section 3, we discuss the realizations of Heisenberg and quantum Heisenberg groups in the deformed operators D and X and the relations between the representations of Heisenberg and quantum Heisenberg algebras and the q-analogous Hermite polynomials. The last section is devoted to some concluding remarks.

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2. The operators D, X and q-series

Recently the q-analogous exponential function $\exp_q(x)$ was applied in the construction of the q-analogous Glauber states [4]. The explicit definition of $\exp_q(x)$ is

$$\exp_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} \tag{1}$$

where $[n]_q!=[n]_q[n-1]_q\ldots[2]_q[1]_q$, and $[x]_q=(q^x-q^{-x})/(q-q^{-1})$. In fact, the q-exponential function is nothing but the eigenfunction of the q-differential operator D, just as the ordinary exponential function $\exp(x)$ is the eigenfunction of the differential operator $\partial=\partial/\partial x$. D is defined via its action on an arbitrary function $f(x)$ [5]

$$Df(x)=\frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})x}. \tag{2}$$

As stressed in [6, 7], when $f(x)$ is analytic, D has a realization in the ordinary differential operator ∂. The D operator has been applied in the studies of quantum groups and their representations [5, 8], and has proved to be a useful tool. To gain more information about this operator, it is necessary to know about its canonical conjugate, X. In this section, we give the explicit expression of X, and verify that it is the canonical conjugate of D. The parameter q that appears in both X and D will be called the deformation parameter in this paper, which is a non-zero real number. When $q\to 1$, $X\to x$ and $D\to\partial$.

Let us begin with introducing a function $\xi(\gamma,x)$ as follows

$$\xi(\gamma,x)\equiv\frac{\sinh(\gamma x)}{x\sinh\gamma}=\sum_{n=0}^{\infty}\frac{\gamma^{2n}x^{2n}}{(2n+1)!}\frac{\gamma}{\sinh\gamma} \tag{3}$$

where $\gamma=\log q$. It is obvious that $\xi(\gamma,x)$ and its inverse $(\xi(\gamma,x))^{-1}$ are both well defined for arbitrary values of γ and x. In the following, we will deal frequently with the operator-valued function

$$\eta_q\equiv\xi(\gamma,x\partial)=\frac{\sinh(\gamma x\partial)}{(x\partial)\sinh\gamma}=\frac{[x\partial]_q}{x\partial} \tag{4}$$

and its inverse. For x being an arbitrary real number, η_q and η_q^{-1} are well defined.

Now we are in a position to introduce a pair of canonical-conjugate operators D and X,

$$D\equiv\frac{1}{x}[x\partial]_q=\partial\cdot\eta_q \tag{5}$$

$$X\equiv\eta_q^{-1}x.$$

Obviously

$$\begin{aligned} &D\xrightarrow{q\to 1}\partial\\ &X\xrightarrow{q\to 1}x. \end{aligned} \tag{6}$$

It is important to note that

$$\begin{aligned} &DX=\partial\cdot x\\ &XD=x\cdot\partial \end{aligned} \tag{7}$$

and

$$[D, X] = [\partial, x] = 1. \tag{8}$$

Therefore the algebra generated by the operators D, X and XD is isomorphic to the differential operator algebra generated by ∂, x and $x\partial$ (the Heisenberg algebra).

It is easy to see from its action on an analytic function $f(x)$ that D is a difference operator with step $(q-q^{-1})x$, satisfying (2). X is a q-analogous 'coordinate' operator and the canonical conjugate of D. The remarkable property of X is that it is keenly related to the q-series, an interesting mathematical object.

To see this point, let us observe the properties of the operator-valued function $F(X)$. For the sake of simplicity, we assume that $F(x)$ is an analytic function defined over $\mathbb{R}$ (the field of real numbers). $F(x)$ can always be expanded into Taylor series:

$$F(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \tag{9}$$

where c_n are coefficients in C. We define the operator-valued $F(X)$ as

$$F(X) = \sum_{n=0}^{\infty} \frac{c_n}{n!} X^n. \tag{10}$$

We are interested in the actions of $F(X)$ on an arbitrary analytic function $f(x)$ defined on $\mathbb{R}$

$$F(X) \cdot f(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} X^n \cdot f(x) \tag{11}$$

and especially the case of $f(x) = 1$,

$$F(X) \cdot 1 = \sum_{n=0}^{\infty} \frac{c_n}{n!} X^n \cdot 1. \tag{12}$$

If one notices that

$$X^n \cdot 1 = \frac{n!}{[n]_q!} x^n \tag{13}$$

it is easy to see that

$$F(X) \cdot 1 = \sum_{n=0}^{\infty} \frac{c_n}{[n]_q!} x^n. \tag{14}$$

As an example, we look into the operator-valued functions $\exp(\alpha X)$, $\sinh(\alpha X)$ and $\cosh(\alpha X)$ for $\alpha \in \mathbb{R}$, especially its action on 1,

$$\begin{aligned} \exp_q(\alpha x) &\equiv \exp(\alpha X) \cdot 1 = \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{[n]_q!} \\ \sinh_q(\alpha x) &\equiv \sinh(\alpha X) \cdot 1 = \sum_{n=0}^{\infty} \frac{(\alpha x)^{2n+1}}{[2n+1]_q!} \\ &= \tfrac{1}{2}(\exp_q(\alpha x) - \exp_q(-\alpha x)) \\ \cosh_q(\alpha x) &\equiv \cosh(\alpha X) \cdot 1 = \sum_{n=0}^{\infty} \frac{(\alpha x)^{2n}}{[2n]_q!} \\ &= \tfrac{1}{2}(\exp_q(\alpha x) + \exp_q(-\alpha x)). \end{aligned} \tag{15}$$

The Biedenharn exponential function [4] $\exp_q(x)$ naturally appears, which is nothing but the eigenfunction of the difference operator D, i.e.

$$D \exp_q(\alpha x) = \alpha \exp_q(\alpha x). \tag{16}$$

It is interesting to see that $\sin_q(\alpha x) = \mathrm{i} \sinh_q(-\mathrm{i}\alpha x)$ is no longer periodic; here we give five of its zero-points:

$$0, \pm\pi[3]_q\left(1 \pm \sqrt{1 - \frac{4[2]_q}{3[3]_q}}\right)\Big/2.$$

Similar to the differential operator realizations of the semisimple Lie algebras, we have new realizations of such algebras *via* the operator D and its conjugate X. Suppose that there is a matrix representation $\{M_n\}$ for the semisimple Lie algebra A; then the new realization is simply

$$(D_1, D_2, \ldots, D_n) M_n \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}. \tag{17}$$

As an example, we give such a realization for the SU(2) algebra explicitly

$$L_+ = X_1 D_2 \qquad L_- = X_2 D_1 \qquad L_3 = \tfrac{1}{2}(X_1 D_1 - X_2 D_2). \tag{18}$$

The representations ρ_j are constructed as

$$|\widehat{j, m}\rangle = \frac{X_1^{j+m} X_2^{j-m}}{(j+m)!(j-m)!} \qquad (m = -j, -j+1, \ldots, j). \tag{19}$$

When the generators act in this space, we have

$$\begin{aligned} L_\pm|\widehat{j, m}\rangle &= (j \pm m + 1)|\widehat{j, m \pm 1}\rangle \\ L_3|\widehat{j, m}\rangle &= m|\widehat{j, m}\rangle. \end{aligned} \tag{20}$$

It should be stressed that the above relations hold when both their sides act on an arbitrary analytic function $f(x_1, x_2)$. In particular, when $f(x_1, x_2)$ is a constant, we have the j-states

$$|j, m\rangle = |\widehat{j, m}\rangle \cdot 1 = \frac{x_1^{j+m} x_2^{j-m}}{[j+m]_q![j-m]_q!} \tag{21}$$

which seem to be identical to the j-states for the $SU_q(2)$ quantum algebra [9, 10]. Of course, the actions of the generators on the states are still of a classical type

$$\begin{aligned} L_\pm|j, m\rangle &= (j \pm m + 1)|j, m \pm 1\rangle \\ L_3|j, m\rangle &= m|j, m\rangle. \end{aligned} \tag{22}$$

3. The q-Hermite polynomial and the $H_q(4)$ and $H(4)$ algebras

Now we consider a system with the following Hamiltonian

$$\hat{H}_q = -\frac{1}{2} D^2 + \frac{1}{2} X^2 = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \hat{V} \tag{23}$$

where $\hat{V}$ should be understood as a non-local (pseudo-) potential. It is obvious that when $q \to 1$, we have

$$\hat{H}_q = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 \tag{24}$$

which is the Hamiltonian for the simple harmonic oscillator system. If we introduce the operators

$$\begin{aligned} a &\equiv \frac{1}{\sqrt{2}}(D+X) \\ a^\dagger &\equiv \frac{1}{\sqrt{2}}(-D+X) \\ N &\equiv a^\dagger a \end{aligned} \tag{25}$$

then it is easy to check that $\hat{H}_q = N + \frac{1}{2}$ and a, $a^\dagger$ and N form the Heisenberg algebra

$$\begin{aligned} [a, a^\dagger] &= 1 \\ [N, a] &= -a \\ [N, a^\dagger] &= a^\dagger. \end{aligned} \tag{26}$$

Therefore the Hamiltonian can be diagonalized.

It is interesting to note that this system is formally identical to the ordinary system of simple harmonic oscillator, with each state being the Hermite polynomial times a Gaussian function (both operator-valued), i.e.

$$\hat{\psi}_n(X) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-X^2/2} H_n(X) \tag{27}$$

and

$$H_n(X) \equiv \sum_{n=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2X)^{n-2k} \tag{28}$$

where $[n/2]$ is the biggest positive integer less than $n/2$. Therefore the formal Schrödinger equation reads

$$\hat{H}_q \hat{\psi}_n(X) = E_n \hat{\psi}(X). \tag{29}$$

It should be noticed that $\hat{\psi}(X)$ are not true solutions to this Hamiltonian system (we call them the pseudo-solutions), because that X is not a coordinate, but an operator. The exact meaning of the above equation is that it holds when the both sides of the equation act on an arbitrary analytic function $f(x)$, i.e.

$$\hat{H}_q \hat{\psi}(X) \cdot f(x) = E_n \hat{\psi}(X) \cdot f(x) \tag{30}$$

and $\hat{\psi}(X) \cdot f(x)$ are at all the solutions to the system and the Hamiltonian depends on the details of $f(x)$. To avoid ambiguities, we choose the simplest case of $f(x) =$ constant

$$\tilde{\psi}_n(x) \equiv \hat{\psi}_n(X) \cdot 1 = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-X^2/2} \cdot \tilde{H}_n(x) \tag{31}$$

where $\tilde{H}_n(x)$ is just the q-analogous Hermite polynomial,

$$\tilde{H}_n(x) \equiv \sum_{n=0}^{[n/2]} \frac{(-1)^k n!}{k![n-2k]_q!}(2x)^{n-2k}. \tag{32}$$

It is not difficult to check that the q-analogous Hermite polynomial satisfies the following q-analogous recursive relations

$$\begin{aligned} D\tilde{H}_n(x) &= 2n\tilde{H}_{n-1}(x) \\ \tilde{H}_{n+1}(x) &= 2X\cdot \tilde{H}_n(x) - 2n\tilde{H}_{n-1}(x) \\ \tilde{H}_n(-x) &= (-1)^n\tilde{H}_n(x). \end{aligned} \tag{33}$$

The Schrödinger equation for this system is

$$\hat{H}\tilde{\psi}_n(x) = E_n\tilde{\psi}_n(x). \tag{34}$$

Now we turn to the system of Q-deformed oscillator [4, 11-14] with quantum group $\mathrm{H}_Q(4)$ symmetry. The $\mathrm{H}_Q(4)$ algebra has been shown phenomenologically to be a good candidate for the symmetry of vibrating diatomic molecules [16]. The Hamiltonian for this system is

$$H = \tfrac{1}{2}(A_Q A_Q^\dagger + A_Q^\dagger A_Q) \tag{35}$$

where we have introduced the Q-analogous creation and annihilation operators

$$A_Q = a\sqrt{\xi(\Gamma, N)} \qquad A_Q^\dagger = \sqrt{\xi(\Gamma, N)}a^\dagger \qquad Q = \mathrm{e}^\Gamma \tag{36}$$

and

$$N = a^\dagger a = -\tfrac{1}{2}D^2 + \tfrac{1}{2}X^2 - \tfrac{1}{2}. \tag{37}$$

One can show without any difficulty that A_Q, $A_Q^\dagger$ and N give the realization of the Q-oscillator algebra

$$\begin{aligned} [A_Q, A_Q^\dagger] &= [N+1]_Q - [N]_Q \\ [N, A_Q^\dagger] &= A_Q^\dagger \\ [N, A_Q] &= -A_Q \end{aligned} \tag{38}$$

which is a quantum algebra with the co-product, co-unit and antipodal mapping well defined [6]. According to [4, 11-15], the infinite dimensional representation of the Q-oscillator algebra is isomorphic to that of the simple harmonic oscillator algebra in (31), i.e.

$$\psi_n(X)\cdot 1 = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\mathrm{e}^{-X^2/2}H_n(X)\cdot 1. \tag{39}$$

The Schrödinger equation is

$$H(X)\psi_n(X)\cdot 1 = \tilde{E}_n\psi_n(X)\cdot 1 \tag{40}$$

where the energy spectrum is $\tilde{E}_n = \frac{1}{2}([n]_Q + [n+1]_Q)$. The pseudo-wavefunctions are descendent states from the pseudo-vacuum state $\psi_0(X)\cdot 1$ with the vacuum defined by $A_Q\psi_0(X)\cdot 1 = 0$, and

$$\psi_n(X) = \frac{(A_Q^\dagger)^n}{\sqrt{[n]_Q!}}\psi_0(X). \tag{41}$$

The actions of the generators yield

$$\begin{aligned} A_Q\psi_n(X)\cdot 1 &= \sqrt{[n]_Q}\psi_{n-1}(X)\cdot 1 \\ A_Q^\dagger\psi_n(X)\cdot 1 &= \sqrt{[n+1]_Q}\psi_{n+1}(X)\cdot 1 \\ N\psi_n(X)\cdot 1 &= n\psi_n(X)\cdot 1. \end{aligned} \tag{42}$$

For Q not a root of unity, the properties of the above solutions to the system are like those to the linear oscillator system [9, 13], i.e. every Fock state can be raised or lowered to a higher or lower state by the actions of the q-analogous creation or annihilation operators.

The higher excitation states are

$$\tilde{\psi}_n(x)=\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,\mathrm{e}^{-x^2/2}\,\tilde{H}_n(x). \tag{43}$$

Therefore the Hilbert space of this Q-deformed system is

$$F=\{\tilde{\psi}_n(x),\, n=0,1,2,\ldots\} \tag{44}$$

and the actions of the generators yield

$$\begin{aligned} A_Q\tilde{\psi}_n(x)&=\sqrt{[n]_Q}\,\tilde{\psi}_{n-1}(x)\\ A_Q^{\dagger}\tilde{\psi}_n(x)&=\sqrt{[n+1]_Q}\,\tilde{\psi}_{n+1}(\,)\\ N\tilde{\psi}_n(x)&=n\tilde{\psi}_n(x). \end{aligned} \tag{45}$$

However, if Q is a root of unity, strange properties may appear. Suppose that p is the least positive integer such that $Q^p=\pm 1$, then $[p]_Q=0$. One will encounter singularities while composing the representations of the Q-deformed oscillator algebra, or the solutions to the system [14]. The representation is

$$V=\left\{|n\rangle=\frac{(A_Q^{\dagger})^n}{\sqrt{[n]_Q!}}|0\rangle,\ n=0,1,2,\ldots\right\}. \tag{46}$$

When $p=1$, the Q-deformed oscillator is identical to the linear oscillator. If p is a positive integer greater than 1, however, the Fock space constructed above is decomposed into infinite invariant subspaces

$$V=\bigcup_{l=0}^{\infty} V_p^l \tag{47}$$

where

$$V_p^l=\left\{|pl+n\rangle=\frac{(A_Q^{\dagger})^n}{\sqrt{[n]_Q!}}|pl\rangle,\ n=0,1,2,\ldots,p-1\right\}. \tag{48}$$

In the particular case of $p=2$, we get the complete irreducible Fock spaces for the Q-oscillator as follows

$$V_2^l=\{|2l\rangle, |2l+1\rangle\} \qquad l=0,1,2,\ldots. \tag{49}$$

And it can be checked easily that in the space V_2^l, the operators for the Q-oscillator satisfy the following relations

$$[A_Q, A_Q^{\dagger}]_+=1 \qquad (A_Q^{\dagger})^2=(A_Q)^2=0. \tag{50}$$

4. Concluding remarks

In this paper, we explicitly constructed the representations of the Lie and quantum Heisenberg algebras via the new operators D and X, which are formal functions of operator X. We obtained functions of coordinate x and the extra parameter q (for quantum groups, there is another independent Q, the deformation parameter) through their actions on an arbitrary analytic function. These q-series-like functions are polynomials of finite terms or series of infinite terms with the factorials in the expansions replaced by q-number factorials such as $n!$ replaced by $[n]_q!$, etc. This is a systematic way to get q-series and their analogous recursive relations and other properties from the ordinary special functions or series.

We know that quantum groups are potential dynamical symmetries in some physical systems involving the violations of the perfect Lie symmetries. Among such systems, we single out the well known Heisenberg XXZ spin chain model [3, 5], where the deviation of Z from X induces the violation of SU(2) symmetry, and the newly proposed quantum group theoretic approach to vibrating and rotating diatomic molecules, where the rigid rotational and linear vibrational symmetries SU(2) and H(4) are violated [16]. These violations are shown to be described (for the latter phenomenologically up to now) by the quantum group symmetries. One of the essential guarantees that allows us to do so is the fact that the systems in which quantum group symmetries are realized and the systems in which Lie group symmetries are realized correspond to the same eigenfunctions but a different energy spectrum.

However, it is interesting to notice that, in the present approach, the eigenstates are different from the ordinary case but possess the same energy spectrum. The new results of this paper may bring new possibilities to these studies.

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